HIGHER ORDER SOFTWARE, INC.
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LEVEL II

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WHAT FUZZY HOS MAY MEAN

by

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WHAT "FUZZY H.O.S." MAY MEAN

Henri M. Prade*, Lucia M. Vaina*

ABSTRACT

The intended objective of this paper is the investigation of the possible fuzzy extensions of H.O.S. methodology. After of brief recall of this methodology and a detailed presentation of the fuzzy concepts which are needed, the notion of fuzzy data type is introduced and discussed, along with its consequences for control maps. The general question of (fuzzy) reliability is then dealt with.

I. INTRODUCTION

Higher Order Software (HOS) has been developed by Magaret Hamilton and Saydean Zeldin [4] as a methodology for the specification of reliable software systems. During the time of this development, fuzzy set theory - initiated by Lotfi Zadeh [12] fourteen years ago - has received more and more attention from researchers, and many valuable contributions to fuzzy system theory have appeared. The reader interested in general monographies about fuzzy sets and systems is addressed to the books by Negoita and Ralescu [10], Kaufmann [9] and Dubois and Prade [3].

However, no work has yet been published dealing with the specification of fuzzy systems. Our paper is presented as an initial attempt to investigate this problem. Although HOS is not the only existing methodology for the specification of large computer-based systems, it seems that, based on the notion of function, HOS is a strong candidate to begin investigation into this problem.

After a presentation of the basic theoretical constructs used in HOS - data types, functions and control structures - several important notions of fuzzy set theory, such as fuzzy function, are reintroduced and discussed. Different kinds of fuzzy data types, as well as their consequences for control maps, are then described. Finally, the problem of reliability of fuzzy systems is considered.

Before beginning, it must be emphasized that "fuzzy specification of systems" and specification of fuzzy systems" do not mean the same thing. "Fuzzy specification" can be described as "approximate" or even "incomplete specification" (which is not a good means of obtaining a reliable system), while "specification of fuzzy systems" refers to the specification of systems which have a behavior only approximately known or which are too complex to be described in great detail.

II. MAIN CHARACTERISTICS OF HOS

The specification of computer-based systems in HOS is done independently of implementation. The software system is formalized in terms of three theoretical constructs:

- data types
- functions/operations
- control maps.

A data type, roughly speaking, represents the kind of entities that the system under consideration manipulates, or, in other words, operates on or produces. A type of value, though, has no meaning by itself. The set of operations which are usually performed and the set of axioms which these operations must satisfy determine the meaning of the data type.

Thus data types of algebraically specified in HOS. A data type consists of [8]:

- a set of objects, called its members;
- a set of functions, called its primitive operations, whose domains and ranges are specified. Either the domain or the range of each primitive operation must include (possibly within some Cartesian product) their own set of members of the data type; and
- a set of axioms which describe the way the primitive operation interact with one another and perhaps with other functions.

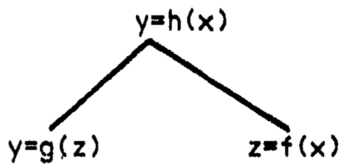
The concept of data type has been extensively discussed by Cushing in [1]. Several examples are provided in [1], such as "Stack" or "Time".

Moreover, each data type has, as a member, an ideal element, namely REJECT, which is the output of any operation when it has no genuine output of the expected sort (i.e., when some members of the domain of the operation are not ampped onto members of the range). In AXES [6], which is a Specification Language based on HOS methodology, six very general data types are prespecified: Boolean, Property, Set, Natural Numbers, Integer and Rational Number. They are the intrinsic types of AXES. The functions (or operations) are thus the entities that operate on or produce members of data types.

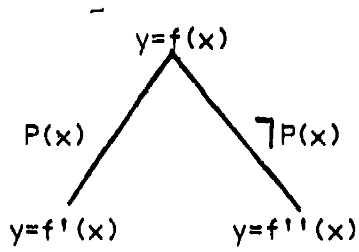
Control structures determine how a function relates to those into which it is decomposed. Obviously only non-primitive operations can be decomposed by means of control maps. If a desired function on a type cannot be defined in terms of primitives, it must be added to the operation specification of the type, thus creating a new data type. AXES, however, allows us to write functions (which are representable as control maps) also as derived operations: they are then characterized by assertions that specify their interactive behavior with other functions (which have been already characterized).

There are three primitive control structures which are used in HOS: (see [8],[1])

- composition

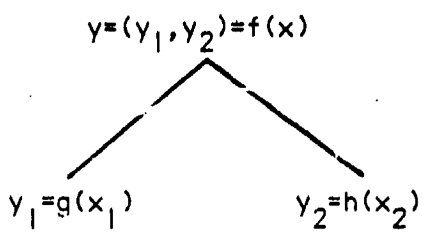


- set partition:



where P is a predicate defined on the domain of f. (P and ¬ P induce a partition of this domain.)

- class partition:



with $x=(x_1,x_2)$

A system whose control structure is decomposable in terms of these three primitive control structures is guaranteed to be reliable in the sense that the six axioms on which HOS is based [4] are then satisfied. (These axioms, which must not be confused with the axioms of a given data type are derived from a set of empirical data and considerations.) AXES [6] is able to express a specification in a form which is equivalent to an HOS control map. A representative application of HOS methodology may be found in [7], where a very complex system in analyzed and specified.

III. FUZZY SETS AND FUZZY FUNCTIONS

Because HOS methodology attempts to define a system as a function, the concept of fuzzy function seems to be necessary to deal with any fuzzy extension of HOS. Surprisingly, there are very few published works devoted to this topic in the fuzzy set literature. Some of the ideas presented here have explicitly appeared for the first time in [3] (which is not available yet).

After a brief recall of the difference between fuzzy set and possibility distribution and of the notion of fuzzy relation, three general kinds of mappings, gathered under the label "fuzzy functions" are introduced. The composition of these different fuzzy functions is described. The notion of the fuzzy partition of a set is then presented.

1. Fuzzy Set versus Possibility Distribution. Fuzzy Relation.

Let X be a universe and A be a fuzzy set on X. A is viewed as a subset of X without precise boundaries. For all $x \in X$, $\mu_A(x)$ is the degree of membership of x to A. The fuzzy set A is thus defined by the pairs

$$(x,\ \mu_A(x)),\quad x \in X \quad .$$

It is assumed here that the valuation set is the interval [0,1], i.e., $\mu_A(x) \in [0,1]$. Let v be a variable which takes its values on X. If E is a non-fuzzy subset of X, to say that v is an element of E indicates that any element in E could possibly be a value of v. In this way, the statement "v is an element of E" induces a possibility distribution π over X which associates to each $x \in X$ the possibility $\Pi(v=x)$ that x is that value of v.

$$\Pi(v=x)=\pi(x)=\begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$$

This point of view may be extended to a fuzzy set A in the following way:

$$\Pi(v=x)=\Pi(x)=\mu_A(x) \quad .$$

Thus a fuzzy set induces a possibility distribution. A is said to be a fuzzy restriction on the (possible) values of v. The concepts of possibility and of possibility distribution, which must not be confused with those of probability and probability distribution (there is no additivity axiom in possibility theory), have been recently introduced by Zadeh in [14].

Let us consider an example. X is the real line $\mathbb{R}$. A fuzzy number $\tilde{m}$ on $\mathbb{R}$ is a fuzzy set on $\mathbb{R}$, such that:

- its membership function $\tilde{m}$ is continuous;
- $\tilde{m}$ is normalized: $\exists\, m \in \mathbb{R},\ \mu_{\tilde{m}}(m)=1$;
- $\tilde{m}$ is convex: $\forall\, x \in \mathbb{R},\ \forall\, z \in \mathbb{R},\ \forall\, y \in [x,z],\ \mu_{\tilde{m}}(y) \geq \min(\mu_{\tilde{m}}(x),\mu_{\tilde{m}}(z))$.

$\tilde{m}$ may be viewed either as a fuzzy set of real numbers clustered around m, or as a possibility distribution on the value of some ill-known quantity.

A fuzzy relation R on the Cartesian product $X \times Y$ is a fuzzy set on $X \times Y$:

$$((x,y),\ \mu_R(x,y)),\quad x \in X,\ y \in Y \ .$$

$\mu_R(x,y)$ may be viewed as expressing the strength of the link from x to y.

2. Fuzzy Functions / Fuzzy Operations.

Classically a function f is a many-one correspondence between two sets, namely the domain X and the range Y or the function: For all $x \in X$, there exists at most one $y \in Y$ such that $y=f(x)$. If there are several y such the $y=f(x)$, f is just a relation.

"Fuzzy function" can be understood in several ways according to where fuzziness lies. From an interpretative point of view, three main types of fuzzy functions may be considered:

- ordinary functions having fuzzy properties or satisfying fuzzy constraints;
- functions which just have fuzzy arguments and fuzzy values, without being fuzzy themselves; they are ordinary functions between sets of fuzzy sets.
- ill-known functions which even map non-fuzzy arguments on fuzzy values. From now on, the expression "fuzzy function" will be used only for this last kind.

Hybridation between these three basic types are possible.

N.B. Functions which map fuzzy arguments on non-fuzzy values, for instance, the function

$$(\text{fuzzy parts of } \mathbb{R}): \tilde{\mathcal{P}}(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$\tilde{m} \longrightarrow \inf\{m, \mu_{\tilde{m}}(m)=1\}$$

are not considered in the following. These functions "defuzzify".

The three kinds of functions related to fuzziness are now reviewed.

α) Fuzzily constrained functions:

This kind of function was first considered by Negoita and Ralescu [10].

Let f be an ordinary function from X to Y. Let A and B be two fuzzy sets of X and Y respectively. A and B are referred to respectively as the fuzzy domain and the fuzzy range of f if and only if:

(1) $\forall\ x \in X$, such that $f(x)$ is defined, $\mu_B[f(x)] \geq \mu_A(x)$.

Let us consider an example (taken from [3]):

"Big trucks must go slowly.": X is the set of trucks, Y is a set of speeds, f assigns a speed limit f(x) to each truck x. A are the fuzzy sets of big trucks and low speeds respectively. The above inequality means: "The bigger the truck, the lower its speed limit." It is shown in [3] that proverbs having the same syntactic structure as the above regulation may be also modelled by functions with fuzzy domain and fuzzy range.

The composition of such functions is very easy. Let g be a function from Y to Z with a fuzzy domain B and a fuzzy range C such that $\mu_C[g(y)] \geq \mu_B(y)$ where $y \in Y$. Then $g \circ f$ is a function from X to Z with a fuzzy domain A and a fuzzy range C since

$$\mu_B[f(x)] \geq \mu_A(x), \quad \mu_C[g(y)] \geq \mu_B(y) \quad \text{and } y=f(x)$$

entail

$$\mu_C[(g \circ f)(x)] \geq \mu_A(x) \ .$$

N.B. More generally, the composition of g with f yield a function with a fuzzy domain and a fuzzy range as soon as the fuzzy range B of f is included in the fuzzy range B' of g, i.e., $\mu_{B'}(y) \geq \mu_B(y)$ with $y=f(x)$.

Binary operations are particular cases of functions. They are mappings from a Cartesian product $X \times X$ to a set Y (possibly equal to X). Let us denote by * such an operation. Before Negoita and Ralescu [10] considered the above notion of fuzzily constrained function, Rosenfeld [11] had introduced the following definitions, when Y=X:

- A fuzzy set A of X is closed under * if and only if

(2) $\forall\ x_1 \in X,\ \forall x_2 \in X,\ \mu_A(x_1 * x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$

which is exactly the same as the inequality (1) for the fuzzily constrained functions, as the Cartisian product $A \times A$ is defined as

$$((x_i, x_j),\ \min(\mu_a(x_i), \mu_A(x_j))),\quad x_i \in X,\quad x_j \in X.$$

- If (X,*) is a group, A is a fuzzy subgroup of X if and only if the above inequality (2) is satisfied and $\forall x \in X$, $\mu_A(x^{-1}) \geq \mu_A(x)$ holds, where $x^{-1}x = e$ and e is the identity.

The last inequality means the mapping $x \longrightarrow x^{-1}$ is a fuzzily constraint function. By symmetry it entails $\mu_A(x^{-1}) = \mu_A(x)$.

N.B. If X is a real Euclidean space, let us consider the operation $\perp_\lambda$: $x_1 \perp_\lambda x_2 = \lambda x_1 + (1-\lambda)x_2$ with $\lambda \in [0,1]$. We see then that the condition of convexity of a fuzzy number $\tilde{m}$ is nothing but the fact that $\tilde{m}$ is closed under $\perp_\lambda$, $\forall\ \lambda \in [0,1]$.

Remarks:

1. The same approach allows us to define fuzzy properties such as fuzzy injectivity, fuzzy surjectivity, fuzzy continuity for an ordinary function (see [3]). Let us give, for instance, a possible definition of the fuzzy surjectivity: f from X to Y is said to be fuzzily surjective on the fuzzy set B of Y if and only if

(3) $\forall\ \ y \in Y,\ \exists\, x \in X,\ \mu_R(y, f(x)) \geq \mu_B(y)$

where μ_R is a fuzzy relation of proximity on Y (i.e., reflexive: $\mu_r(y,y) = 1$, $\forall y \in Y$, and symmetrical :

$$\mu_R(y, y') = \mu_R(y', y),\ \forall\ y \in Y,\ \forall\ y' \in Y).$$

(3) means "the more y belongs to B, the closer is a neighbor of y which has an antecedent x in the sense of f" ([3]).

2. Instead of considering a function f with a fuzzy domain A and a fuzzy range B, it may be interesting to deal with functions having just a fuzzy domain A on X and no constraint on the range Y. However, f is still fuzzily constrained in the sense of (1) because $\mu_Y(y)=1 \geq \mu_A(x)$ with $y=f(x)$. Then the image of A by f is the fuzzy set: $\left\{\left(y, \sup_{x, y=f(x)} \mu_A(x)\right), y \in Y\right\}$.

If $\nexists$ x such that y=f(x), then the membership degree of y to the fuzzy image of A is zero. $\mu_A(x)$ may be interpreted as the degree to which f is applicable to x.
Similarly if f has an ordinary domain X and a fuzzy range B on Y ((1) no more holds), where $\mu_B(y)$ is interpreted as a degree of "acceptability" of y,
$\{(f(x), \mu_B(f(x))), x \in X\}$ is the acceptable image of X by f.

If f has a fuzzy domain A and a fuzzy range B, the acceptable image of A by f is the fuzzy set (using the operator "min" for the intersection of fuzzy sets):

$$\left\{\left(y, \min\left[\sup_{x, y=f(x)} \mu_A(x), \mu_B(y)\right]\right), y \in Y\right\},$$

which is equal to (if (1) holds):

$$\left\{\left(y, \sup_{x, y=f(x)} \mu_A(x)\right), y \in Y\right\}$$

i.e., the image of A by f is acceptable.

Lastly, if the fuzzy domain (or the fuzzy range) is defined as a fuzzy set of ordinary sub-domains (or subranges), say $A_1, \ldots, A_m$:

$$A = \{(\mu(i), A_i), i=1,m\} \qquad A_i \subset X$$

then A may be reduced as

$$A = \left\{\left(x, \max_{i, x \in A_i} \mu(i)\right), x \in X\right\}$$

or $$A = \left\{\left(x, \min_{i, x \in A_i} \mu(i)\right), x \in X\right\}$$ for examples.

3) Ordinary functions between two sets of fuzzy sets:

Let f be an ordinary function from X to Y. This function may be extended as a function F from $\tilde{\mathcal{P}}(X)$ to $\tilde{\mathcal{P}}(Y)$ $\tilde{\mathcal{P}}(X)$ denotes the ordinary set of the fuzzy sets of X). The membership function $\mu_{\tilde{y}}$ of the fuzzy set $\tilde{y}=F(\tilde{x})$, where $\tilde{x}$ denotes a fuzzy set of X, is given by the extension principle (see Zadeh [13]:

$$(4) \qquad \mu_{\tilde{y}}(y)=\mu_{F(\tilde{x})}(y)=\sup_{x\in f^{-1}(y)} \mu_{\tilde{x}}(x)$$

$$=0 \quad \text{if } f^{-1}(y)=\phi$$

where $f^{-1}(y)$ is the set of antecedents of y.

(4) is not formally different from the definition of the image by f of its fuzzy dom[a]in A, but here the point of view is different. We are no longer interested [illegible] [gl]obal image of a domain, but rather in the images of fuzzy points, fuzzy [illegible] [o]f the domain of f. $\tilde{x}$ may be interpreted as a possibility distribution of val[ues] possibly clustered around some mean value.

Th[e] [illegible]iction of F to X is f. Moreover, it should be noticed that the image of a [f]uzzy s[illegible]ton (λ,x) is $(\lambda,f(x))$, where λ denotes a membership value.

[illegible] of the operation *, (4) is written:

$$(5) \qquad \mu_{\tilde{y}}(y)=\mu_{\tilde{x}\circledast\tilde{x}'}(y)=\sup_{x,x',y=x*x'} \min(\mu_{\tilde{x}}(x),\mu_{\tilde{x}'}(x'))$$

where $\circledast$ is the operation from $\tilde{\mathcal{P}}(X)\times\tilde{\mathcal{P}}(X)$ to $\tilde{\mathcal{P}}(Y)$, induced from $*$. The addition of fuzzy numbers can be defined using (5), for instance. (See Dubois and Prade, [2],[3]).

The composition of two extended functions F from $\tilde{\mathcal{P}}(X)$ to $\tilde{\mathcal{P}}(Y)$ and G from $\tilde{\mathcal{P}}(Y)$ to $\tilde{\mathcal{P}}(Z)$ respectively is the extension H of the composition of the original functions f and g.

Proof:

$$\mu_{G(F(\tilde{x}))}(z)=\mu_{G(\tilde{y})}(z)=\sup_{y\varepsilon g^{-1}(z)}\mu_{\tilde{y}}(y)=\sup_{y\varepsilon g^{-1}(z)}\mu_{F(\tilde{x})}(y)=$$

$$\sup_{y\varepsilon g^{-1}(z)}\ \sup_{x\varepsilon f^{-1}(y)}\mu_{\tilde{x}}(x)=\sup_{x\varepsilon(g.f)^{-1}(z)}\mu_{\tilde{x}}(x)=\mu_{H(\tilde{x})}(z) \qquad \text{Q.E.D.}$$

It should be clear that all the ordinary functions from $\tilde{\mathcal{P}}(X)$ to $\tilde{\mathcal{P}}(Y)$ are not extensions of the ordinary functions from X to Y.

γ) Fuzzy function of a non-fuzzy variable [3]

Two points of view will be successively developed:

- the image of $x\varepsilon X$ is a fuzzy set $\tilde{f}(x)$ of Y .
- x is mapped in Y through a fuzzy set of functions.

1) Jiggling functions:

A jiggling (or fuzzifying) function from X to Y is an ordinary function from X to $\tilde{\mathcal{P}}(Y)$:

$$\tilde{f} : x \longrightarrow \tilde{f}(x) \ .$$

The image of x is blurred by the "jiggling" of the function. $\tilde{f}(x)$ may be also of as a possibility distribution of a non-fuzzy image of x.

The notion of jiggling function is equivalent to that of fuzzy relation. $\tilde{f}$ is associated with a fuzzy relation in R such that:

(6) $\forall\, x\varepsilon X,\ \forall\, y\varepsilon Y,\ \ \mu_{\tilde{f}(x)}(y)=\mu_{R(x,y)}$

The composition of jiggling functions is defined by

(7) $\mu_{(\tilde{g}\circ\tilde{f})(x)}(z)=\sup_{y\varepsilon Y}\min\,(\mu_{\tilde{f}(x)}(y),\mu_{\tilde{g}(y)}(z))$

where $\tilde{g}$ is a jiggling function from Y to Z.

This composition may be interpreted as follows: given an intermediary point y, the membership of $z\varepsilon Z$ in $(\tilde{g}\circ\tilde{f})(x)$ is bounded by the memberships of y in $\tilde{f}(x)$ and z in $\tilde{g}(y)$. The membership of z in $(\tilde{g}\circ\tilde{f})(x)$ corresponds to the best intermediary point.

This composition is different from the classical composition of functions, as $\tilde{f}$ and $\tilde{g}$, being functions from X to $\tilde{\mathcal{P}}(Y)$ and from Y to $\tilde{\mathcal{P}}(Z)$, cannot be composed together.

A fuzzy binary operation on X is a jiggling function from X×X to $\tilde{\mathcal{P}}(X)$.

Remarks:

1. Jiggling functions may have fuzzy domain A and fuzzy range B in the sense of α):

$$(8)\qquad \sup_{x\in X} \min\,(\mu_{\tilde{f}(x)}(y),\mu_A(x)) \leq \mu_B(y), \quad \forall\, y.$$

When $\tilde{f}(x)$ is non fuzzy, i.e., $\exists !\, y^*$, $\mu_{\tilde{f}(x)}(y^*)=1$ and $\forall\, y\neq y^*$ $\mu_{\tilde{f}(x)}(y)=0$, (8) gives back (i).

Using (7), one may prove that if $\tilde{g}$ has B as a fuzzy domain and C as a fuzzy range, then $\tilde{g}^{\circ}\tilde{f}$ has A as a fuzzy domain and C as a fuzzy range. Negoita and Ralescu [10], who considering (8) in the framework of fuzzy relations, also introduced fuzzy relations (i.e., jiggling functions) such as:

$$(9)\qquad \mu_R(x,y)=\mu_{\tilde{f}(x)}(y)\leq \min\,(\mu_A(x),\mu_B(y)) \quad .$$

(The link between x and y cannot be stronger than the degrees of membership of x and y to the domain and the range respectively.)

If B is normalized, by using (7) it can be proven that jiggling functions with fuzzy domain and fuzzy range in the sense of (9) can also be composed.

2. Jiggling functions may be extended in the sense of β): $\tilde{f}$, which is a function from X to $\tilde{\mathcal{P}}(Y)$, is extended in $\tilde{F}$, which is a function from $\tilde{\mathcal{P}}(X)$ to $\tilde{\mathcal{P}}(Y)$ by:

$$(10)\qquad \mu_{\tilde{F}(\tilde{x})}(y)= \sup_{x\in X}\ \min\,(\mu_{\tilde{x}}(x),\mu_{\tilde{f}(x)}(y)) \quad .$$

One may verify that

$$\mu_{\tilde{G}(\tilde{F}(\tilde{x}))}(z)= \sup_{x\in X}\ \min\,(\mu_{\tilde{x}}(x),\mu_{(\tilde{g}^{\circ}\tilde{f})(x)}(z)) \quad .$$

This shows that the extension of $\tilde{g}^{\circ}\tilde{f}$ defined by (7) and (11) is nothing more than the ordinary composition of $\tilde{F}$ and $\tilde{G}$, functions from $\tilde{\mathcal{P}}(X)$ to $\tilde{\mathcal{P}}(Y)$ and from $\tilde{\mathcal{P}}(Y)$ to $\tilde{\mathcal{P}}(Z)$ respectively.

Lastly, it must be pointed out that there ordinary functions from $\tilde{\mathcal{P}}(X)$ to $\tilde{\mathcal{P}}(Y)$ which are not extended jiggling (or not jiggling) functions). See [3].

ii) Fuzzy bunch of functions

A fuzzy bunch $\mathcal{F}$ of ordinary functions from X to Y is a fuzzy set of Y^X: each function f from X to Y has a membership degree $\mu_{\mathcal{F}}(f)$ in $\mathcal{F}$. A jiggling function f may be viewed as a fuzzy bunch (see [3]). A fuzzy bunch, though, is not canonically reducible to jiggling functions, since there may be two functions f and g from X to Y such that:

$$\exists\, x,\ f(x)=g(x)=y \quad \text{with } \mu_{\mathcal{F}}(f)\neq\mu_{\mathcal{F}}(g).$$

The degree of membership of y to the fuzzy image of x by $\mathcal{F}$ may then be ambiguous. This can never happen with a jiggling function as the membership degree of y to $\tilde{f}(x)$ is uniquely defined by $\mu_{\tilde{f}(x)}(y)=\mu_R(x,y)$. A fuzzy bunch may be considered as a multimodal fuzzy relation.

Let $\mathcal{F}$ and $\mathcal{G}$ be two fuzzy bunches from X to Y and from Y to Z respectively. The composition $\mathcal{H}=\mathcal{F}\circ\mathcal{G}$ is defined by

$$\text{(11)} \qquad \mu_{\mathcal{H}}(h)=\sup_{\substack{f,g\\ h=g\circ f}} \min\left(\mu_{\mathcal{F}}(f),\mu_{\mathcal{G}}(g)\right) \quad .$$

(11) generalizes (7).

3. Partition and Fuzzy Sets.

For the sake of simplicity, only partitions in two parts will be considered.

Let A be a fuzzy set of X and P an ordinary predicate on X. P induces a partition on X: the set of elements $x\in X$ such that P(x) is true and the set of elements $x\in X$ such that $\neg P(x)$ is true. In the same way A is partitioned in A_1 and A_2 such that

$$\mu_{A_1}(x)=\begin{cases}\mu_A(x) & \text{if } P(x) \text{ is true}\\ 0 & \text{otherwise}\end{cases} \quad ; \quad \mu_{A_2}(x)=\begin{cases}\mu_A(x) & \text{is } \neg P(x) \text{ is true}\\ 0 & \text{otherwise}\end{cases} \quad .$$

Let us now consider the fuzzy partition of a set, namely X. Let $\tilde{P}$ be a fuzzy predicate. The truth-value, $v(\tilde{P}(x))$ of $\tilde{P}(x)$ belongs to $[0,1]$. Thus $\tilde{P}$ induces a fuzzy set Q:

$$\mu_Q(x)=v(\tilde{P}(x)) \quad .$$

The complement $\overline{Q}$ of Q is

$$\mu_{\overline{Q}}(x)=1-v(\tilde{P}(x))=v(\neg \tilde{P}(x)) \quad .$$

Q and $\overline{Q}$ are overlapping, and $(Q,\overline{Q})$ is not an ordinary partition of X, but rather only a fuzzy one:

$$\mu_{Q\cap\overline{Q}}(x)=\min\ (\mu_Q(x),1-\mu_Q(x))\neq 0 \qquad \text{if } \mu_Q(x)\neq 1 \text{ or } 0$$

$$\mu_{Q\cup\overline{Q}}(x)=\max\ (\mu_Q(x),1-\mu_Q(x))\neq 1 \qquad \text{if } \ _Q(x)\neq 1 \text{ or } 0 \quad .$$

N.B. In the sense of the operators $\boldsymbol{\cap}$ and $\boldsymbol{\cup}$ of bold intersection and bold union:

$$\mu_{A\,\boldsymbol{\cap}\,B}(x)=\max\ (0,\mu_A(x)+\mu_B(x)-1)$$

$$\mu_{A\,\boldsymbol{\cup}\,B}(x)=\min\ (1,\mu_A(x)+\mu_B(x))$$

$(Q,\overline{Q})$ is a partition: $Q\,\boldsymbol{\cap}\,\overline{Q}=\phi$; $Q\,\boldsymbol{\cup}\,\overline{Q}=X$.

4. Concluding Remarks about Fuzzy Functions and Operations.

There are basically two kinds of functions or operations involving fuzziness:

- Those functions and operations which operate on and produce fuzzy sets, values or numbers in a non-fuzzy manner: e.g.,
  - the extension in the sense of 2.β. of any function from $\mathbb{R}$ to R ;
  - the operations of union, intersection, complementations on fuzzy sets;
  - the addition and the multiplication of fuzzy numbers.

• The fuzzy functions and fuzzy operations which, operating in a <u>fuzzy</u> manner on fuzzy or non-fuzzy quantitites, produces fuzzy ones. The jiggling functions and the fuzzy bunches of functions introduced in 2.γ. are of that kind: e.g.,

- a function which increases x slightly:

$$\mathbb{R} \longrightarrow (\mathbb{R})$$
$$x \longmapsto x \oplus \tilde{\Sigma} = \tilde{y}$$

where $\tilde{\Sigma}$ is a small fuzzy number and $\oplus$ is the symbol of the extended addition. $\tilde{y}$ is the possibility distribution of results got from x by increasing it slightly.

- a fuzzy operation on $\mathbb{R}$: to approximately take the arithmetic mean of two real numbers x and y:

$$x*y = x \odot \tilde{\lambda} \oplus y(1 \ominus \tilde{\lambda}) = \tilde{z}$$

where $\odot$ denotes the extended multiplication of fuzzy numbers and $\ominus$ the extended substraction. $\tilde{\lambda}$ is a fuzzy number such that $\mu_{\tilde{\lambda}}(1/2)=1$. $\tilde{z}$ is a possibility distribution of values clustered around $(x+y)/2$.

Most of the functions involving fuzziness introduced in V.2. have been considered in the framework of category theory. For instance, functions considered in 2.α. satisfying (1), in 2.γ. with (8), and in 2.γ. satisfying (9), correspond to the morphisms of categories Set(L), $\text{Set}_g(L)$ and $\text{Set}_f(L)$ respectively. Moreover, the dichotomy discussed in this remark corresponds to the difference between:

- ordinary categories of fuzzy sets and
- fuzzy categories.

Two points of view corresponding to fuzzy bunches and jiggling junctions exist concerning fuzzy categories. See Negoita and Ralescu [11] for details. Also see [3] for a short survey.

Although the link between categories and HOS representation has been recently emphazised (see "A Note on Arrows and Control Structures: Category Theory and HOS" by S. Cushing in [8]), it has seemed preferable to present here the functions involving fuzziness in a more elementary way as sophisticated formalisms are not perhaps suitable in working with concepts at this early stage of development.

IV. HOS METHODOLOGY AND FUZZY SETS

Let us begin with the question: "What is a fuzzy data type?" Naturally there are several kinds of "fuzzy data types", more or less fuzzy, depending on where fuzziness lies.

First, only the members of the data-type may be fuzzy entities. The primitive and derived operations, though, are ordinary ones. The axioms are also of a classical nature. We have, for instance, the data-type "Fuzzy Set" (which is different from the data-type "Set"), which is also a data-type in the ordinary sense. It corresponds to the non-fuzzy manipulation of fuzzy quantities. Control structures are always the same.

A little more fuzzy gives the concept of fuzzy rejection. Fuzzy rejection corresponds to the ordinary functions equipped with a fuzzy domain A on X (see V.2.). It means that all the elements of X are not equally suitable as inputs of the function under consideration. Thus each x is rejected with a degree equal to $1-\mu_A(x)$ (if $\mu_A(x)=0$ the output of the function is the classical REJECT). The fact that a function has a fuzzy domain is compatible with the set partition control structure (see V .3).

Already intrinsically fuzzy are the data types where some primitive operations or functions are fuzzy (i.e. are jiggling functions in the sense of V .2.β). The members may or may not be fuzzy. The axioms are classical ones: it is not because an operation is fuzzy that it interacts in a fuzzy manner with itself or other ones. Two examples of such operations were given in III.4 . Jiggling functions/operations maybe viewed as ordinary functions, symbolically from X to (Y). It is as jiggling functions from X to Y, though, that these functions are composed by formula (7). Formally, we have the abstract control structure:

$$\begin{tikzcd}[column sep=small] & \tilde{z}=\tilde{h}(x) \arrow[dl, dash] \arrow[dr, dash] & \\ \tilde{z}=\tilde{g}(y) & & \tilde{y}=\tilde{f}(x) \end{tikzcd}$$

(7) may be formally rewritten as

$$H(x,z)= \sup_{y \in Y} \min (G(y,z),F(x,y))$$

which may be decomposed in ordering control maps provided that we have the primitive operations "min", and "sup" of a given function on a given domain.

Lastly, the axioms of a data type may be themselves fuzzy, even if the operations or/and the members are not fuzzy. What is a fuzzy axiom? It is a fuzzy description of the way operations are allowed to interact with one another. The description is fuzzy because the allowed interaction is supposedly ill-known. For instance, we may say that the operation * is approximately commutative if A*B is approximately equal to B*A for all A and B where "approximately equal" is modelled by a fuzzy relation. * is approximately commutative in another sense if A*B=B*A for <u>most</u> of the A and B. Fuzzy axioms are linguistic statements on the allowed interactions of the operations. Linguistic statements can be represented using possibility distributions - see Zadeh's PRUF [18]. It seems possible to develop fuzzy mathematics by starting with fuzzy axioms and using approximate reasoning (see Zadeh [17]). Fuzzy mathematics is fuzzy manipulation of entities and is completely different from mathematics of fuzzy sets, which is simply classical mathematics (ordinary manipulation) of "new" entities - the fuzzy sets. In the framework of elementary geometry, Zadeh, in Part C of [15], has given some examples of "fuzzy mathematics".

Thus, there are several kinds of fuzzy data types as members, operations, and even axioms may be fuzzy independently of each other.

V. <u>FUZZY RELIABILITY</u>

What is classical reliability? Let us quote Margaret Hamilton and Saydean Zeldin [5]:

> "For a reliable system is a predictable system; it does exactly what it is intended to do. In attempting to define a system as a function, we already have incorporated an element of reliability in that we assert that for every value of "x" we expect to produce one and only one value for "y". That is, we expect the system to predictably produce the same result each time we apply f [y=f(x)] to a particular value."

It is clear that the functions involving fuzziness which have been presented in V.2.α and V.2.β are reliable in the above classical sense.

Obviously, it is not true for the jiggling functions or the fuzzy bunches. Let us consider a jiggling function $\tilde{f}$ from X to Y. The image of x by $\tilde{f}$ is $\tilde{y}=\tilde{f}(x)$. If we regard $\tilde{y}$ as a fuzzy set (i.e. we consider $\tilde{f}$ rather as an ordinary function from X to $\tilde{\wp}(Y)$), $\tilde{f}$ is reliable in the ordinary sense:

$$\forall \; x \in X, \; \exists ! \tilde{y} \text{ (or } \nexists \tilde{y}\text{) such that } \tilde{y}=\tilde{f}(x).$$

If $\tilde{y}$ is viewed as a possbility distribution on the value of the non-fuzzy output y of $\tilde{f}$ when the input is x, though, $\tilde{f}$ is no more absolutely reliable, as we have only an approximate idea (modelled by $\tilde{y}$) of the more or less possible outputs of f when the input is x. It is from this point of view that the jiggling functions are composed by (7). It is then only possible to speak of fuzzy reliability as the behavior of the function is only approximately known.

VI. CONCLUDING REMARKS

HOS methodology seems to be a good guideline to imagining what may be the specification of fuzzy systems. Even if some of the concepts presented in section V seem totally removed from logic as we apply it today, we should keep in mind that people, on the whole, tend to cling to many preconceived ideas concerning the structure of mathematics and how to use it to solve the problems of today. If we can open-mindedly look at these problems with fuzzy concepts and approaches within our sights, we might be surprised to see that complex and difficult questions may be easier to grasp when presented in a fuzzy way. Obviously, this is particularly true in the field of soft sciences.

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REFERENCES

1. Cushing, Steven, "Algebraic specification of data types in higher order software (HOS)", Tech. Report #13, Higher Order Software, Inc., Cambridge, MA. Also in Proc. 11th Hawaii International Conference on Systems Sciences, 1978.

2. Dubois, Didier & Prade, Henri, "Operations on fuzzy numbers", Int. J. of Systems Science, Vol. 9, no. 6, pp. 613-626, 1978.

3. Dubois, Didier & Prade, Henri, "Fuzzy sets and systems". To appear published by Academic Press, 1979.

4. Hamilton, Margaret & Zeldin, Saydean, "Higher Order Software - A methodology for defining software", IEEE Trans.on Soft. Eng., Vol. 2, no. 1, pp. 9-32, 1976.

5. Hamilton, Margaret & Zeldin, Saydean, "Reliability in terms of predictability", Tech. Repor. #16, Higher Order Software, Inc., Cambridge, MA, 1978.

6. Hamilton, Margaret, Zeldin, Saydean, Cushing, Steven & Heath, W., "AXES syntax description", Tech. Report #4, Higher Order Software, Inc., Cambridge, MA, 1976.

7. H.O.S., "The application of HOS to PLRS", Tech. Report #12, Higher Order Software, Inc., Cambridge, MA, 1977.

8. H.O.S., "Axiomatic Analysis", Tech. Report #19, Higher Order Software, Inc., Cambridge, MA, 1978.

9. Kaufmann, Arnold, "Introduction a la theorie des sous ensembles flous", 5 volumes, Masson and Cie, Paris, 1973-1978.

10. Negoita, C.V. & Ralescu, D.A., "Applications of fuzzy sets to systems analysis", ISR 11, Birkhauser Verlag, 1975.

11. Rosenfeld, A., "Fuzzy groups", J. of Math.Anal. and Appl., Vol. 35, pp. 512-517, 1971.

12. Zadeh, Lotfi A., "Fuzzy sets", Info. & Cont., Vol. 8, pp. 338-353, 1965.

13. Zadeh, Lotfi A., "The concept of linguistic variable and its application to approximate reasoning", Inf. Sci., Part 1, Vol. 8, pp. 199-243. Part 2, Vol. 8, pp. 301-357. Part 3, Vol. 9, pp. 43-80, 1975.

14. Zadeh, Lotfi A., "Fuzzy sets as a basis for a theory of possibility", Int.J. for Fuzzy Sets and Systems, Vol. 1, no. 1, pp. 3-28, 1978.

15. Zadeh, Lotfi A., "A theory of approximate reasoning (AR)", in "Machine Intelligence", Vol. 19, Michie ed., American Elsenier, New York, 1978.

16. Zadeh, Lotfi A., "PRUF: A meaning representation language for natural languages", Memo UCB/ERL, 77161, Berkeley, California, 1977.